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# Optimal Control of Nonlinear Systems: A Recursive Approach

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**Abstract**—In this paper, we shall generalize our result [1] on the optimal control of bilinear systems to nonlinear systems. Adomian's decomposition is used to derive series expansions of the optimal control and state.

**Keywords**—Nonlinear systems, Optimal control, Adomian's decomposition.

## 1. INTRODUCTION

The theory of optimal control of nonlinear systems has always been an active area of research; see [2–7] to cite a few. However, the methods used led to difficult computations. In a recent paper [1], we solved successfully and in a simple manner the bilinear quadratic optimal control problem. The idea was to transform the original optimal control problem into a recursive optimization problem, so that at each step we obtain a term of a series representation of the optimal control and a term of the corresponding series representation of the optimal state, i.e., $u^* = \sum_{k\geq 0} u_k^*$, $x^* = \sum_{k\geq 0} x_k^*$. These series were shown to be absolutely and uniformly convergent on $[0,T]$, for any fixed but arbitrary $T > 0$, to the solution of the original problem.

Continuing our effort to provide an effective means to compute the optimal control, we propose in this paper to generalize the results in [1] to the optimal control of nonlinear systems.

The outline of the paper is as follows: in Section 2, we shall derive a functional series representation of the state in terms of the functions involved in the series representation of the input. We shall prove that if the series representation of the input is absolutely and uniformly convergent, so is the series representation of the state. In Section 3, we state and solve the optimal control of nonlinear systems.

## 2. THE FUNCTIONAL EXPANSION FOR NONLINEAR INPUT-OUTPUT MAPS

In this section, we shall present a functional expansion for nonlinear i/o maps suitable for the computation of the optimal control.

Consider the nonlinear system defined by

$$\begin{aligned} \frac{dx}{dt} &= f(x,u), \qquad t \in [0,T], \\ x(0) &= x_0, \end{aligned} \tag{1}$$

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX where $f$ is an analytic function from $\mathbf{R}^q \times \mathbf{R}^p$ to $\mathbf{R}^q$ such that $f(0,0) = 0$, $u$ is a $p$-dimensional input vector, and $x$ is a $q$-dimensional state vector.

We shall use Adomian's decomposition method [8] (see also [1,9]) and seek a series expansion of the state of (1) in the form

$$x(t) = \sum_{n\geq 0} x^{[n]}(t), \tag{2}$$

while

$$u(t) = \sum_{k\geq 0} u^{[k]}(t). \tag{3}$$

Thus,

$$f(x,u) = f\left(\sum_{n\geq 0} x^{[n]}, \sum_{k\geq 0} u^{[k]}\right) = F_0\left(x^{[0]}\right) + \sum_{m\geq 1} F_m\left(x^{[0]},\dots,x^{[m]},u^{[0]},\dots,u^{[m-1]}\right), \tag{4}$$

where

$$\begin{aligned} &F_0\left(x^{[0]}\right) = f\left(x^{[0]},0\right), \\ &F_m\left(x^{[0]},\dots,x^{[m]};u^{[0]},\dots,u^{[m-1]}\right) = \\ &\sum_{p=1}^{m} \sum_{\substack{k_1+\cdots+k_p=m \\ k_1\geq 1\cdots k_p\geq 1}} f_p\left(x^{[0]},0\right)\left(x^{[k_1]},u^{[k_1-1]};\dots;x^{[k_p]},u^{[k_p-1]}\right), \qquad m\geq 1, \end{aligned} \tag{5}$$

where the $f_p(x^{[0]},0)(.,.;\dots;.,.)$ are the $p$-linear maps appearing in the Taylor's series expansion of $f$ around $(x^{[0]},0)$,

$$f(x,u) = f\left(x^{[0]},0\right) + \sum_{p\geq 1} f_p\left(x^{[0]},0\right)\left(x-x^{[0]},u;\dots;x-x^{[0]},u\right). \tag{6}$$

REMARK 1. The $F_m$ are polynomials in $x^{[1]},\dots,x^{[m]},u^{[0]},\dots,u^{[m-1]}$, $m\geq 1$.

Going back to equation (1), replacing $x$, $u$, $f(x,u)$ by their respective series expansions, we get

$$\frac{d}{dt}\sum_{n\geq 0} x^{[n]} = F_0\left(x^{[0]}\right) + \sum_{m\geq 1} F_m\left(x^{[0]},\dots,x^{[m]},u^{[0]},\dots,u^{[m-1]}\right). \tag{7}$$

We shall define the sequence $\{x^{[n]}\}_{n\geq 0}$ by

$$\begin{aligned} &\frac{d}{dt}x^{[0]} = f\left(x^{[0]},0\right), && x^{[0]}(0) = x_0, \\ &\frac{d}{dt}x^{[m+1]} = F_{m+1}\left(x^{[0]},\dots,x^{[m+1]},u^{[0]},\dots,u^{[m]}\right), && x^{[m+1]}(0) = 0, \qquad m\geq 0, \end{aligned} \tag{8}$$

and make the following assumption.

ASSUMPTION A.

(i) $\|B\| \leq \beta$.
(ii) $\exists\, L_1, C_1, C_2 > 0/\ |f(x,u) - \sum_{i=0}^{l-1} F_i| \leq C_1|x-\overline{x}_{l-1}| + C_2|u-\overline{u}_{l-1}|$, $l\geq L_1+1$, *where* $\overline{x}_l = \sum_{i=0}^{l} x^{[i]}$ *and* $\overline{u}_l = \sum_{i=0}^{l} u^{[i]}$.
(iii) $\widehat{\rho} = \rho C_1 < 1$.

We claim the following theorem.

THEOREM 1. *Let $u \in L^\infty[0,T,\mathbf{R}^p]$ such that it can be expanded in a uniformly and absolutely convergent series as $\sum_{k\geq 0} u^{[k]}$. Let Assumption A be satisfied. Then the state $x$ of the system (1) has an absolutely and uniformly convergent series representation $\sum_{n\geq 0} x^{[n]}$ whose terms are computed recursively using equations (5) and (8).*

PROOF. That the sum $\sum_{n\geq 0} x^{[n]}$ is the solution to equation (1) is trivial. What remains to be proved is the fact that the series $\sum_{n\geq 0} x^{[n]}$ is uniformly and absolutely convergent when $\sum_{k\geq 0} u^{[k]}$ is.

Let $z_l = x - \overline{x}_l$ . We have

$$\frac{d}{dt} z_l = f(x,u) - \sum_{i=0}^{l-1} F_i, \qquad z_l(0) = 0. \tag{9}$$

A straightforward application of Lemma (2.1) in [1] and Gronwall's Lemma, together with Assumption A, will yield the result (proof along the proof of Theorem (2.1) in [1]).

## 3. OPTIMAL CONTROL OF NONLINEAR SYSTEMS

In this section, we shall consider the optimization problem

$$\min_u J =: h(x)|_{t=T} + \int_0^T g(x,u)\, dt \tag{10}$$

subject to the constraint (1), where $h$ and $g$ are any convex analytic functions.

Let $J_l$ denote the value of the cost functional when $x$ is replaced by $\overline{x}_{l+1}$ and $u$ by $\overline{u}_l$, that is,

$$J_l =: h(\overline{x}_{l+1})|_{t=T} + \int_0^T g(\overline{x}_{l+1}, \overline{u}_l)\, dt. \tag{11}$$

Suppose that the optimum values $u^{[0]},\dots,u^{[l-1]}$ and the corresponding $x^{[1]},\dots,x^{[l]}$ have been obtained from the minimization of $J_0,\dots,J_{l-1}$ with respect to $u^{[0]},\dots,u^{[l-1]}$ (under corresponding constraint), respectively. We shall seek the minimum of $J_l$ with respect to $u^{[l]}$ subject to the constraint

$$\begin{aligned} &\frac{d}{dt} x^{[l+1]} = F_{l+1}\left(x^{[0]},\dots,x^{[l+1]},u^{[0]},\dots,u^{[l]}\right), \\ &x^{[l+1]}(0) = 0, \qquad l \geq 0. \end{aligned} \tag{12}$$

We have

$$F_{l+1}\left(x^{[0]},\dots,x^{[l+1]};u^{[0]},\dots,u^{[l]}\right) = \frac{\partial f}{\partial x}\left(x^{[0]},0\right) x^{[l+1]} + \frac{\partial f}{\partial u}\left(x^{[0]},0\right) u^{[l]} + \widehat{F}_{l+1}, \qquad l \geq 0, \tag{13}$$

where

$$\widehat{F}_{l+1} = \sum_{p=2}^{l+1} \sum_{\substack{k_1+\dots+k_p=l+1 \\ k_1\geq 1 \cdots k_p \geq 1}} f_p\left(x^{[0]},0\right)\left(x^{[k_1]},u^{[k_1-1]};\dots;x^{[k_p]},u^{[k_p-1]}\right), \tag{14}$$

$\widehat{F}_{l+1}$ is independent of $x^{[l+1]}$ and $u^{[l]}$.

The Hamiltonian of the problem is

$$H_l = g\left(\overline{x}_{l+1},\overline{u}_l\right) + \lambda_l^T \left\{\frac{\partial f}{\partial x}\left(x^{[0]},0\right) x^{[l+1]} + \frac{\partial f}{\partial u}\left(x^{[0]},0\right) u^{[l]} + \widehat{F}_{l+1}\right\}; \tag{15}$$

therefore, the necessary conditions for optimality are

$$\frac{\partial}{\partial u^{[l]}} H_l = 0, \qquad \frac{\partial}{\partial x^{[l+1]}} H_l + \frac{d\lambda_l^T}{dt} = 0, \tag{16}$$

together with the boundary condition

$$\lambda_l^T = \frac{\partial h}{\partial x}(\overline{x}_{l+1}), \qquad \text{at } t = T, \tag{17}$$

which read

$$\frac{\partial g}{\partial u}(\overline{x}_{l+1}, \overline{u}_l) + \lambda_l^T \frac{\partial f}{\partial u}\left(x^{[0]}, 0\right) = 0, \tag{18}$$

$$\frac{\partial g}{\partial x}(\overline{x}_{l+1}, \overline{u}_l) + \lambda_l^T \frac{\partial f}{\partial x}\left(x^{[0]}, 0\right) + \frac{d\lambda_l^T}{dt} = 0. \tag{19}$$

Therefore, locally, around $(\overline{x}_l, \overline{u}_{l-1})$, we have

$$u^{[l]} = -R_l^{-1}\left\{W_l^T x^{[l+1]} + \left(\frac{\partial f}{\partial u}\left(x^{[0]}, 0\right)\right)^T \lambda_l\right\}, \tag{20}$$

$$\frac{d\lambda_l}{dt} = -\left(\frac{\partial f}{\partial x}\left(x^{[0]}, 0\right)\right)^T \lambda_l - Q_l x^{[l+1]} - W_l u^{[l]}, \tag{21}$$

together with

$$\lambda_l = P_l x^{[l+1]}, \qquad \text{at } t = T, \tag{22}$$

where

$$\begin{aligned} R_l &= \frac{\partial^2}{\partial u^2} g(\overline{x}_l, \overline{u}_{l-1}), \\ W_l &= \frac{\partial^2}{\partial x \partial u} g(\overline{x}_l, \overline{u}_{l-1}), \\ Q_l &= \frac{\partial^2}{\partial x^2} g(\overline{x}_l, \overline{u}_{l-1}), \\ P_l &= \frac{\partial^2 h}{\partial x^2}(\overline{x}_l), \end{aligned} \tag{23}$$

provided $R_l$ is nonsingular.

REMARK 2. This result agrees with the bilinear quadratic case presented in [1], where $W_l = 0$, $R_l = R$, and $\frac{\partial f}{\partial u}(x^{[0]}, 0) = B + x^{[0]T} N$.

Standard computations are arguments showing that a necessary condition for optimal control is given by the following theorem.

THEOREM 2. *A necessary condition for the uniformly and absolutely convergent series (2) and (3) to be an optimal solution of the optimal control problem (1) and (10) is that their terms are related by*

$$u^{[l]} = -R_l^{-1}\left\{W_l^T + B^T S\right\} x^{[l+1]} - R_l^{-1} B^T v_l \tag{24}$$

*together with (5) and (8), where $S$ is the solution of the Ricatti differential equation*

$$\begin{gathered} \frac{d}{dt} S + S A_l + A_l^T S - S B R_l^{-1} B^T S + \widehat{Q}_l = 0, \\ S = P_l, \qquad \text{at } t = T, \end{gathered} \tag{25}$$

*and $v_l$ is the solution of the differential equation*

$$\begin{gathered} \frac{d}{dt} v_l + \left\{A^T - S B R_l^{-1} B^T - W_l R_l^{-1} B^T\right\} v_l + S \widehat{F}_{l+1} = 0, \\ v_l = 0, \qquad \text{at } t = T, \end{gathered} \tag{26}$$

*where $R_l, W_l, Q_l, P_l$ are defined in (23) and $A, B, A_l$, and $\widehat{Q}_l$ are defined below:*

$$A = \frac{\partial f}{\partial x}\left(x^{[0]}, 0\right), \tag{27}$$

$$B = \frac{\partial f}{\partial u}\left(x^{[0]}, 0\right), \tag{28}$$

$$A_l = A - BR_l^{-1}W_l^T, \tag{29}$$

$$\widehat{Q}_l = Q_l - W_l R_l^{-1} W_l^T, \tag{30}$$

*and provided $R_l$ is nonsingular for all $l \geq 1$.*

PROOF. The proof goes along the proof of Theorem (3.2) in [1].

## 4. CONCLUSION

In this paper, we have extended our result [1] concerning the quadratic bilinear optimal control to the optimal control of nonlinear systems. The method used is promising and simple to implement—it consists of solving a matrix Ricatti differential equation and linear differential equation at each step. The method is more systematic compared to *ad hoc* techniques presented, for example, in [3]. We shall see in a future paper that the control and state given by (2),(3) and satisfying (24)–(30) is indeed optimal, and see how well this approach compares with existing methods and present extensive numerical simulations.

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